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THE SINE-GORDON, KLEIN-GORDON, AND KORTEWEG-DE VRIES EQUATIONS

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Abstract—The Sine-Gordon, Klein-Gordon, and Kortweg-DeVries (KdV) equations are similarly treated using decomposition to obtain general solutions in terms of initial/boundary conditions when the physically correct conditions are known.

INTRODUCTION

We briefly treat three well-known equations of physics by decomposition. If the correct conditions are given, we must obtain physically realistic solutions not otherwise possible. Exact linearization employing transformations of variables is restricted to a small class of equations and usual linearizations of nonlinear problems to make them tractable lead to solutions that may deviate seriously from the desired physical solution. The solution is found as an n-term approximation $\phi_n = \sum_{i=0}^{n-1} u_i$ that converges to u as $n \to \infty$. The initial term u_0 is a function given in terms of initial/boundary conditions, the forcing term, and an invertible linear deterministic operator. The solution $u = \sum_{n=0}^{\infty} u_n$ is a generalized Taylor series about the function u_0. Analytic functions $f(u)$ in the equations are written $f(u) = \sum_{n=0}^{\infty} A_n(u_0, u_1, \dots, u_n)$ where the A_n have been defined to form a generalized Taylor series about $f(u_0)$. The computability and the generally quite rapid convergence, easily seen numerically, make the method efficient and useful providing verifiable analytic solutions to significant problems without linearization, perturbation, closure approximations, or discretization and consequent demands on computational time. It appears reasonable to view the procedure as a quantitative theory of dynamical systems. Possibly it may yield insights in physical problems such as Navier–Stokes and plasma problems. Of course, much can yet be done in carefully defining ranges of applicability — a matter outside the scope of this paper, which proposes only to provide solution in terms of various possible specified conditions for real *physical* problems.

1) *Klein–Gordon Equation* $u_{tt} - \nabla^2 u = g(u)$: Letting $L_t = \partial^2/\partial t^2$, $L_x = \partial^2/\partial x^2$, $L_y = \partial^2/\partial y^2$, $L_z = \partial^2/\partial z^2$, we rewrite this as

$$[L_t - L_x - L_y - L_z]u = g(u)$$

Solving for each linear operator term in turn in accordance with the decomposition method [1], we have

$$\begin{aligned}
L_t u &= [L_x + L_y + L_z]u + g(u) \\
L_x u &= [L_t - L_y - L_z]u - g(u) \\
L_y u &= [L_t - L_x - L_z]u - g(u) \\
L_z u &= [L_t - L_x - L_y]u - g(u)
\end{aligned}$$

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Operate on each equation by the appropriate inverse. (We use indefinite two-fold integrations for the operators defined here. However, for initial conditions, definite integrations from zero or t_0 to t are convenient.) Thus $L_t^{-1}L_t u = u - \alpha_1 - \alpha_2 t$ where the functions α_1, α_2 are determined from the initial/boundary conditions. We deal the same way with the following equations. Now

$$\begin{aligned}
u &= (\alpha_1 + \alpha_2 t) + L_t^{-1}[L_x + L_y + L_z]u + L_t^{-1}g(u)\\
u &= (\alpha_3 + \alpha_4 x) + L_x^{-1}[L_t - L_y - L_z]u - L_x^{-1}g(u)\\
u &= (\alpha_5 + \alpha_6 y) + L_y^{-1}[L_t - L_x - L_z]u - L_y^{-1}g(u)\\
u &= (\alpha_7 + \alpha_8 z) + L_z^{-1}[L_t - L_x - L_y]u - L_z^{-1}g(u)
\end{aligned}$$

Let $u = \sum_{n=0}^{\infty} u_n$ and $g(u) = \sum_{n=0}^{\infty} A_n$ where the A_n are a special set of polynomials defined in [1] evaluated specifically for $g(u)$, which have the property that A_n depends only on components of u from u_0 to u_n. If more than one nonlinear term exists, each would be written in terms of A_n generated specifically for that nonlinearity. In each of the above equations we identify u_0 as the first two terms, e.g., in the first equation

$$u = u_0 + L_t^{-1}(L_x + L_y + L_z)\sum_{n=0}^{\infty} u_n + L_t^{-1}\sum_{n=0}^{\infty} A_n$$

so that

$$u_{n+1} = L_t^{-1}(L_x + L_y + L_z)u_n + L_t^{-1}A_n$$

and so on. The other equations are similarly written. Each can be carried to some n-term approximation or $\phi_n = \sum_{m=0}^{n} u_m$, which is then used to satisfy the given conditions. The resulting ϕ_n approach u for sufficient n. Adomian and Rach have established in a forthcoming paper that with general conditions, each equation is sufficient and we no longer need to consider all of them, adding and dividing by the number of equations. (When u_0 vanishes in any particular equation, $u_{n+1} = 0$ for $n \geq 0$ that the particular equation makes no contribution.)

Thus the equation is solvable for any given initial boundary conditions that are not simply arbitrary, of course, but compatible with the physical problem. Convergence is best seen numerically [1–3].

2) *The Korteweg-de Vries (KdV) equation*:

This equation arises in a number of physical problems associated with waves. It describes unidirectional wave propagation in shallow water or magnetohydrodynamic waves in cold plasma, and so on. It is an important example of a model for a weakly nonlinear weakly dispersive system with no dissipation. The solution of this and the perturbed KdV or nonlinear Schrödinger equations as initial value problems were the first important contributions to the theory of solitons. We consider it in the form:

$$u_t - 6uu_x + u_{xxx} = 0 \tag{1}$$

Write $L_t = \partial/\partial t$ and $L_{xxx} = \partial^3/\partial x^3$. Equation (1) becomes $L_t u - 6uu_x + L_{xxx}u = 0$, or $L_t u + Nu + L_{xxx}u$ if the nonlinear term $-6uu_x$ is denoted by $Nu = -6u(\partial/\partial x)u$. Solving for the linear terms $L_t u$ and the dispersion term $L_{xxx}u$ in turn we have

$$L_t u = Nu - L_{xxx}u \tag{2}$$

$$L_{xxx}u = Nu - L_t u \tag{3}$$

Defining L_t^{-1} as the definite integration operator over $[0,t]$, we operate on both sides of (2) with L_t^{-1} to obtain

$$u = u(x,0) + L_t^{-1}Nu - L_t^{-1}L_{xxx}u \tag{4}$$

Suppose $u(x,0) = f(x)$ is known. Then $u_0 = f(x)$ and $u_{n+1} = L_t^{-1}A_n - L_t^{-1}L_{xxx}u_n$ for $n \geq 0$ with the A_n determined for the particular Nu. Similarly treating equation (3), we get

$$u_0 = A + Bx + Cx^2$$

$$u_{n+1} = L_{xxx}^{-1}A_n - L_{xxx}^{-1}L_t u_n$$

to determine components of u. We can evaluate "constants of integration" for $\phi_1 = u_0$, $\phi_2 = u_0 + u_1, \ldots, \phi_n = \sum_{m=0}^{n} u_m$ as we go along or leave A, B, C unevaluated until we get to some n, then use ϕ_n to satisfy the conditions. It will be simplest to use (4) since only one condition is required — an initial condition — so that $u_0 = u(x,0)$ is known and no further integration constants are needed (L_t^{-1} is a definite integration).

The A_n are the polynomials defined in [1] for the specific nonlinearity $Nu = -6u(\partial/\partial x)u$. When $Nu = f(u)$ is an analytic function, the $\sum_{n=0}^{\infty} u_n$ is a generalized Taylor series about $f(u_0)$ where u_0 is an analytic bounded function in the interval of interest and the $\sum_{n=0}^{\infty} u_n$ is a generalized Taylor series about the function u_0. Thus once the $A_n(u_0, u_1, \ldots, u_n)$ are specified. we can calculate the approximation $\phi_n = \sum_{i=0}^{n-1} u_i$ to desired n. The limit as $n \to \infty$ is u.

The A_n are easily calculated [1,2]:

$$\begin{aligned}
A_0 &= -6\{u_0(\partial/\partial x)u_0\} \\
A_1 &= -6\{u_0(\partial/\partial x)u_1 + u_1(\partial/\partial x)u_0\} \\
A_2 &= -6\{u_0(\partial/\partial x)u_2 + u_1(\partial/\partial x)u_1 + u_2(\partial/\partial x)u_0\} \\
A_3 &= -6\{u_0(\partial/\partial x)u_3 + u_1(\partial/\partial x)u_2 + u_2(\partial/\partial x)u_1 + u_3(\partial/\partial x)u_0\} \\
&\vdots \\
A_n &= -6\{u_0(\partial/\partial x)u_n + \cdots + u_n(\partial/\partial x)u_0\}
\end{aligned} \tag{5}$$

so that we have an explicit procedure for solution given $u(x,0)$.

3) *Sine–Gordon equation* $\partial^2 u/\partial x \partial t = \sin u$:

If we let $L_x = \partial/\partial x$, $L_t = \partial/\partial t$, $N(u) = \sin u$, we can write either

$$L_x L_t u = N(u) \tag{6}$$

$$L_t L_x u = N(u) \tag{7}$$

Operating on (6) first, with L_x^{-1} then with L_t^{-1}, we obtain

$$u = u(x,0) + u(0,t) + L_t^{-1}L_x^{-1}N(u) \tag{8}$$

(We first get $L_t u = L_t u(0,t) + L_x^{-1}N(u)$ then the result in (8) assuming $u(0,0) = 0$.) Now operating on (7) with L_t^{-1}, then with L_x^{-1}, we obtain

$$u = u(0,t) + u(x,0) + L_x^{-1}L_t^{-1}N(u) \tag{9}$$

Adding (8) and (9) and dividing by two to get a single equation for u,

$$u = u(0,t) + u(x,0) + \tfrac{1}{2}(L_x^{-1}L_t^{-1} + L_t^{-1}L_x^{-1})N(u)$$

We assume a decomposition of the desired solution u into components u_n to be determined. i.e., $u = \sum_{n=0}^{\infty} u_n$, with u_0 identified as

$$u_0 = u(0,t) + u(x,0) \tag{10}$$

A nonlinear term, in this case, $N(u) = \sin u$, is expressed by $N(u) = \sum_{n=0}^{\infty} A_n(u_0, u_1, \dots, u_n)$, where the A_n are special polynomials generated for the specific nonlinearity [1,2]. We now have

$$u = u_0 + \tfrac{1}{2}[L_x^{-1}L_t^{-1} + L_t^{-1}L_x^{-1}]\sum_{n=0}^{\infty} A_n \tag{11}$$

where u_0 is given by (10).

For $N(u) = f(u) - \sin u$ we have

$$\begin{aligned}
A_0 &= \sin u_0 \\
A_1 &= u_1 \cos u_0 \\
A_2 &= u_2 \cos u_0 - (u_1^2/2!) \sin u_0 \\
A_3 &= u_3 \cos u_0 - u_2 u_1 \sin u_0 - (u_1^3/3!) \cos u_0 \\
A_4 &= u_4 \cos u_0 + u_3 u_1(-\sin u_0) + (u_2 u_1^{-2}/2!)(-\cos u_0) \\
&\quad + (u_1^4/4!) \sin u_0 \\
A_5 &= u_5 \cos u_0 + u_4 u_1(-\sin u_0) + (u_3 u_1^2/2/!)(-\cos u_0) \\
&\quad + (u_1^3 u_2/3!) \sin u_0 + (u_1^5/5!) \cos u_0
\end{aligned} \tag{12}$$

and so on.

Thus since u_0 is known, equations (10) and (11) provide the solution $u = u_0 + u_1 + \cdots$ where

$$\begin{aligned}
u_0 &= [u(x,0) + u(0,t)] \\
u_1 &= (1/2)[L_t^{-1}L_x^{-1} + L_x^{-1}L_t^{-1}]A_0 \\
u_2 &= (1/2)[L_t^{-1}L_x^{-1} + L_x^{-1}L_t^{-1}]A_1 \\
&\ \vdots \\
u_{n+1} &= (1/2)[L_t^{-1}L_x^{-1} + L_x^{-1}L_t^{-1}]A_n
\end{aligned}$$

with the A_n in (12). Thus the u_n are determinable and we write $\phi_n = \sum_{i=0}^{n-1} u_i$ as an n-term approximation approaching u in the limit as $n \to \infty$. The solution depends, of course, on the given conditions $u(x,0)$, $u(0,t)$.

Final Remarks: Further work will explore specific numerical results for physical conditions of interest.

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